Hereditary Structural Completeness of Weakly Transitive Logics

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Tuesday 7th November, 2023

Our goal

The following problems are equivalent:

- Characterising the 1-transitive modal logics Λ such that $\vdash_\Lambda$ is HSC.
- Characterising the axiomatic extensions of $\vdash_{\mathsf{wK4}}$ which are HSC.
- Characterising the primitive varieties of wK4-algebras.

Some algebraic tools

Corollary

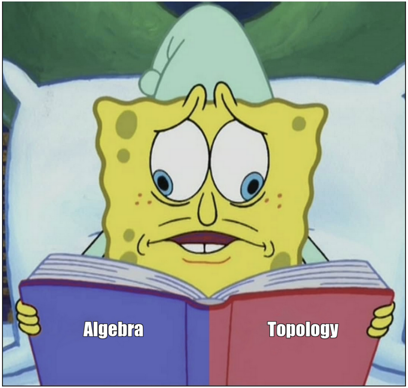
Let K be a primitive variety of 1-transitive algebras. The finite SI members of K are weakly projective in K.

Corollary

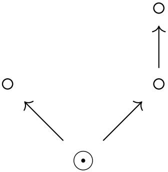
Let K be a variety of 1-transitive algebras such that all its subvarieties have the FMP. If the finite SI members of K are weakly projective in K, then K is primitive.

Jónsson-Tarski duality

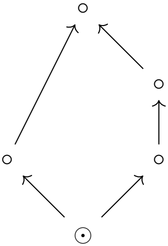
Algebra	Topology
Subalgebra	Quotient space
Quotient	Modal subspace
Finite product	Disjoint union
Finite SI	Finite rooted
Weakly projective	Weakly injective



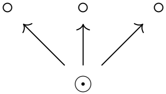
HSC logics above IPC (Citkin)



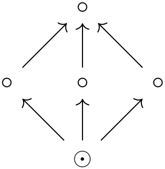
(a) F_5



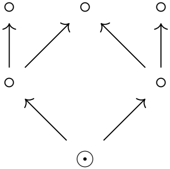
(b) F_6



(c) F_7



(d) F_8



(e) F_9

HSC logics above IPC (Citkin)

Theorem (Citkin)

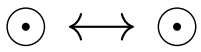
A variety of Heyting algebras is primitive iff it omits F^* for F depicted on the previous slide.

Corollary

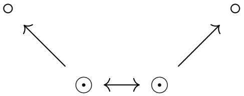
Let Λ be an intermediate logic and let K be the EAS of $\vdash_\Lambda$. TFEA.

- *$\vdash_\Lambda$ is HSC.*
- *K is primitive.*
- *Each frame F depicted on the previous slide is not a Λ-space.*

HSC logics above K4 (Rybakov)



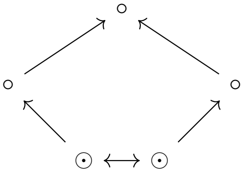
(a) F_1



(b) F_4

(c) F_3

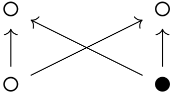
(d) F_{11}



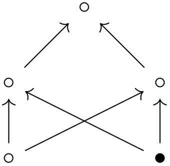
(e) F_{10}

(f) F_2

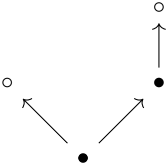
HSC logics above K4 (Rybakov)



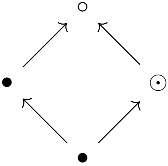
(a) F_{12}



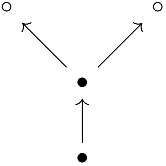
(b) F_{13}



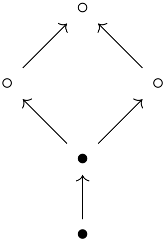
(c) F_{14}



(d) F_{15}

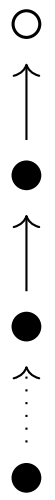


(e) F_{16}



(f) F_{17}

HSC logics above K4 (Carr)



(a) G_n

(b) G_ω

HSC logics above K4 (Carr-Rybakov)

Theorem (Carr-Rybakov)

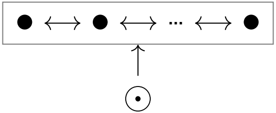
A variety of K4-algebras is primitive iff it omits F^* for F depicted on the previous slides and omits G_n for some $n < \omega$.

Corollary

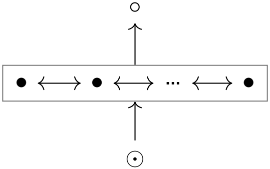
Let Λ be a transitive logic and let K be the EAS of $\vdash_\Lambda$. TFEA.

- $\vdash_\Lambda$ *is HSC.*
- K *is primitive.*
- *Each frame F depicted on the previous slides is not a Λ-space and for some $n < \omega$, G_n is not a Λ-space.*

HSC logics above wK4



(a) I_n



(b) J_n

HSC logics above wK4

Theorem (SL)

A variety of wK4*-algebras is primitive iff it omits F^* for F depicted on the previous slides, omits G_n for some $n < \omega$ and omits I_n and J_n for all $n < \omega$.*

Corollary

Let Λ be a 1-transitive logic and let K be the EAS of $\vdash_\Lambda$. TFEA.

- *$\vdash_\Lambda$ is HSC.*
- *K is primitive.*
- *Each frame F depicted on the previous slides is not a Λ-space, for some $n < \omega$, G_n is not a Λ-space and for all $n < \omega$, I_n and J_n are not Λ-spaces.*

First direction

Corollary

Let K be a primitive variety of 1-transitive algebras. The finite SI members of K are weakly projective in K.

Definition

An algebra A is weakly projective in a variety K if for all $B \in K$, $A \in \mathbb{H}(B)$ implies $A \in \mathbb{IS}(B)$.

Proof strategy

Corollary

Let K be a variety of 1-transitive algebras such that all its subvarieties have the FMP. If the finite SI members of K are weakly projective in K, then K is primitive.

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We want to prove the following.

- Any cobwebby variety has the FMP.
- In a cobwebby variety, the finite SI algebras are weakly projective.

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To achieve that goal

- Describe properties of skeletons of cobwebby spaces (a.k.a. cobwebby skeletons).

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- Identify forbidden configurations in cobwebby spaces.

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To achieve that goal

- Describe properties of skeletons of cobwebby spaces (a.k.a. cobwebby skeletons).
- Identify forbidden configurations in cobwebby spaces.
- Prove main structural theorem.

Properties of cobwebby skeletons

Let X be a cobwebby space.

Lemma

The maximal clusters of X are singletons.

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Either the maximal clusters of X are reflexive or X is an antichain of irreflexive points.

Properties of cobwebby skeletons

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Either the maximal clusters of X are reflexive or X is an antichain of irreflexive points.

Let X be a cobwebby skeleton and $x \in X$ be irreflexive.

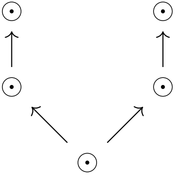
Lemma

Any chain in $R^{-1}[x]$ is finite and any $y \in R^{-1}[x]$ is irreflexive.

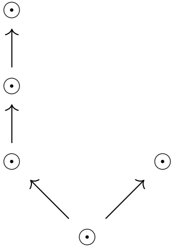
Lemma

Assume that $y \, R \, x$ and $y \, R \, z$. Then x and z are comparable.

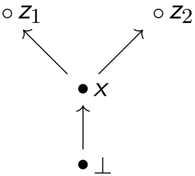
Forbidden configurations



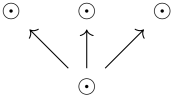
(a) X_2



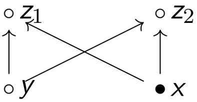
(b) X_3



(c) X_4



(d) X_1



(e) X_5

Main structural theorem

Let A be a cobwebby algebra and X its dual. Assume that A is finitely generated and SI. Then the frame underlying X is a sequential composition $\bigoplus_{\alpha \leq \beta} Q_\alpha$ of finite frames $(Q_\alpha)_{\alpha \leq \beta}$ for some ordinal $\beta = \lambda + n$, with λ a limit ordinal and $n < \omega$, such that the following hold.

- Q_α is a single cluster if $\alpha = \beta$ or α is a limit ordinal.
- Q_α is a single cluster, a two cluster antichain or H if $\alpha = 0$.
- Q_α is a single cluster or a two cluster antichain otherwise.
- Any maximal cluster is a single reflexive point, if Q_α is a two cluster antichain then $Q_{\alpha+1}$ only contains improper clusters, and any non-minimal, proper cluster contains a reflexive point.
- If X contains an irreflexive cluster, then $n \neq 0$ and there is some $m \in \{1, \dots, n\}$ such that for all $\alpha < \lambda + m$, Q_α does not contain any irreflexive cluster, and for all $\alpha \geq \lambda + m$, Q_α is an irreflexive cluster. Moreover, if $m < n$, then $Q_{\lambda+m-1}$ is a single cluster.

Wrapping up

We can prove the following.

- Any cobwebby variety has the FMP.
- In a cobwebby variety, the finite SI algebras are weakly projective.