Hereditary Structural Completeness of Weakly Transitive Logics

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Admissible and derivable rules

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Definition

A deductive system $\vdash$ is hereditarily structurally complete (HSC) if all of its extensions (equivalently, axiomatic extensions) are SC.

1-transitive logics

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A Kripke frame is a wK4-frame if its relation is 1-transitive, i.e.

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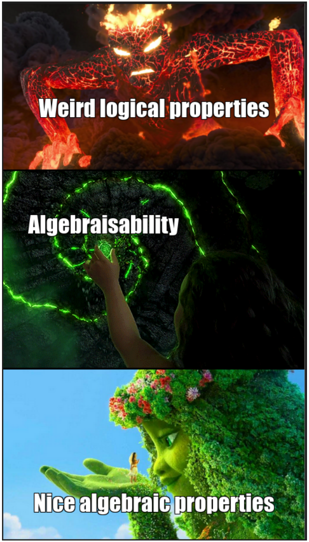
Definition

A modal algebra is a wK4-algebra if it validates $a \wedge \Box a \leq \Box\Box a$ for all a.

Dictionary

Logic	Algebra
Deductive system	EAS (variety)
Rules	Quasi-equations
Admissible	Valid in $F_K(\omega)$
Derivable	Valid in K
Extensions	Subquasi-variety
Axiomatic extensions	Subvarieties
HSC	Primitive



Wrapping up

The following problems are equivalent:

- Characterising the 1-transitive modal logics Λ such that $\vdash_{\Lambda}$ is HSC.
- Characterising the axiomatic extensions of $\vdash_{\mathsf{wK4}}$ which are HSC.
- Characterising the primitive varieties of wK4-algebras.

Subdirectly irreducible and weakly projective algebras

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Lemma

Let K be a primitive variety of finite type. The finite SI members of K are weakly projective in K.

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An algebra A is weakly projective in a variety K if for all $B \in K$, $A \in \mathbb{H}(B)$ implies $A \in \mathbb{IS}(B)$.

Theorem

Let K be a variety with EDPC, and such that all its subvarieties have the FMP. If the finite SI members of K are weakly projective in K, then K is primitive.

Equationally definable principal congruences

Definition

A variety K has equationally definable principal congruences (EDPC) if there is a finite set of equations $\Phi(x, y, z, t)$ such that for all $A \in K$ and $a, b, c, d \in A$, we have

$$(a, b) \in \mathrm{Cg}^A(c, d) \qquad \text{iff} \qquad A \models \Phi(c, d, a, b).$$

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Proposition

If K has EDPC, we can define a finite set of equations $\Phi_n(x_1, y_1, \ldots, x_n, y_n, z, t)$ such that for all $A \in K$ and $a, b, c_1, d_1, \ldots, c_n, d_n \in A$, we have

$$(a, b) \in \mathrm{Cg}^A((c_1, d_1), \ldots, (c_n, d_n)) \qquad \textit{iff} \qquad A \models \Phi_n(a, b, c_1, d_1, \ldots, c_n, d_n).$$

DDT and EDPC

Theorem

If K has EDPC, then

$$\Theta, \phi_1 \approx \psi_1, \ldots, \phi_n \approx \psi_n \models_K \varepsilon \approx \delta$$
$$\textit{iff} \qquad \Theta \models_K \Phi_n(\phi_1, \psi_1, \ldots, \phi_n, \psi_n, \varepsilon, \delta).$$

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Let $\vdash$ be a deductive system with variety K as its EAS. Then $\vdash$ has a DDT iff K has EDPC.

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Theorem

Let $\vdash$ be a deductive system with variety K as its EAS. Then $\vdash$ has a DDT iff K has EDPC.

Corollary

Varieties of 1-transitive algebras have EDPC.

Finite model property

Definition

A variety K has the finite model property (FMP) if for any equation $\varepsilon \approx \delta$ such that $A \not\models \varepsilon \approx \delta$ for some $A \in K$, there is a finite $A \in K$ such that $A \not\models \varepsilon \approx \delta$.

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A variety K has the FMP iff $K = \mathbb{V}(K_{\mathrm{Fin}})$, where K_{Fin} is the class of finite members of K.

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Lemma

Let K be a variety with the FMP and EDPC. Then $K = \mathbb{Q}(K_{\mathrm{FinSI}})$, where K_{FinSI} is the class of finite SI members of K.

Finishing off

Proposition

A variety K is primitive iff for all subvariety L of K, we have $L = \mathbb{Q}(F_L(\omega))$.

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Theorem

Let K be a variety with EDPC, and such that all its subvarieties have the FMP. If the finite SI members of K are weakly projective in K, then K is primitive.

Wrapping up

Corollary

Let K be a primitive variety of 1-transitive algebras. The finite SI members of K are weakly projective in K.

Corollary

Let K be a variety of 1-transitive algebras such that all its subvarieties have the FMP. If the finite SI members of K are weakly projective in K, then K is primitive.

Modal spaces

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Jónsson-Tarski duality

Algebra	Topology
Subalgebra	Quotient space
Quotient	Modal subspace
Finite product	Disjoint union
Finite SI	Finite rooted
Weakly projective	Weakly injective