On Conway's Numbers and Games

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INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Contents

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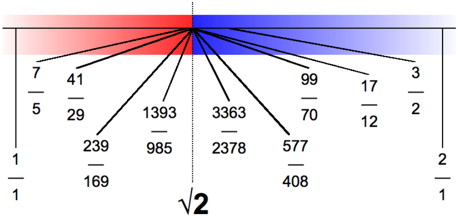

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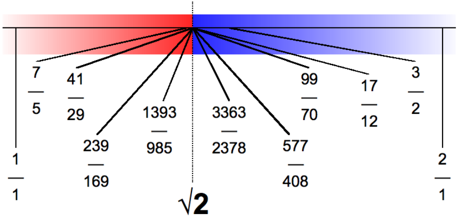

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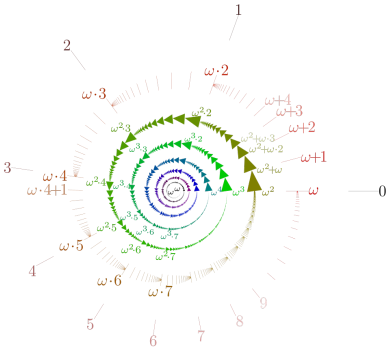

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If L, R are two sets of numbers, and for all $l \in L, r \in R$, we have $l \not\geq r$, then $\{L \mid R\}$ is a number.

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Convention

If $x = \{L \mid R\}$, we write x^L for the typical member of L, x^R for the typical member of R, and $\{x^L \mid x^R\}$ for x itself.
We write No for the (proper) class of numbers.

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If L, R are two sets of numbers, and for all $l \in L, r \in R$, we have $l \ngeq r$, then $\{L \mid R\}$ is a number.

Definitions

We say $x \geq y$ iff $x^R \nleq y$ and $x \nleq y^L$ (for all x^R and y^L).

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Finally, we let

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid \\ x^L y + xy^R - x^L y^R, x^R y + xy^L + x^R y^L\}$$

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$$
\begin{aligned}
-2 &= \{\ \mid -1\} = \{\ \mid -1, 0\} = \{\ \mid -1, 1\} = \{\ \mid -1, 0, 1\} \\
-1 &= \{\ \mid 0\} = \{\ \mid 0, 1\} \\
-\tfrac{1}{2} &= \{-1 \mid 0\} = \{-1 \mid 0, 1\} \\
0 &= \{\ \mid\ \} = \{-1 \mid\ \} = \{\ \mid 1\} = \{-1 \mid 1\} \\
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What is the number $x = \{-1 \mid 2\}$?

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Exercise

Prove that $1 + 1 = 2$ and that $\frac{1}{2} + \frac{1}{2} = 1$.

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What is the number $\{0 \mid \frac{1}{2}\}$? What about $\{\frac{1}{2} \mid 1\}$? $\{1 \mid 2\}$? $\{2 \mid\}$?

Day ω

On day ω, we obtain

- $\{0, 1, 2, \dots \mid \} = \{1, 2, 4, \dots \mid \} = \omega$,

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- $\frac{1}{3} = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64}, \dots \mid \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \frac{1}{2} - \frac{1}{8} - \frac{1}{32}, \dots\}$,

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We also 'recreate' dyadic rationals.

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- $\sqrt{2}, e, \pi$ as cuts of dyadic numbers,

We also 'recreate' dyadic rationals. For example, $\{$dyadic rationals $< \frac{3}{8} \mid$ dyadic rationals $> \frac{3}{8}\}$ turns out to be $\frac{3}{8}$.

Some more Numbers

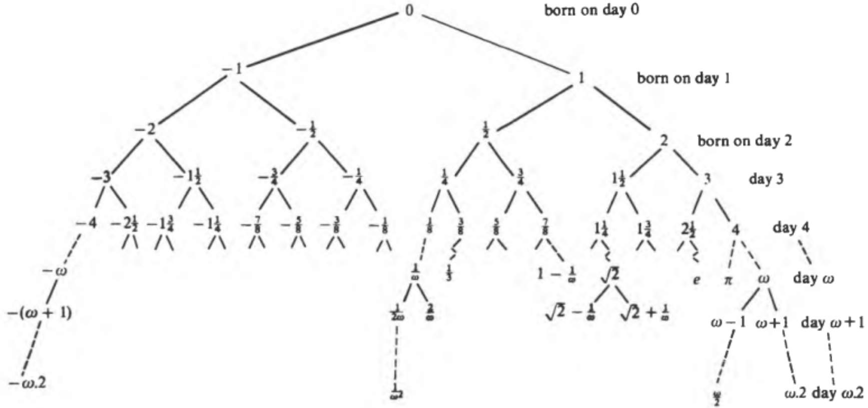

FIG. 0. When the first few numbers were born.

Games

Construction

If L, R are two sets of games, then $\{L \mid R\}$ is a games.

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Two games x and y are *identical* ($x \equiv y$) iff every x^L is identical to some y^L, every x^R is identical to some y^R and vice versa.

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Remark

Multiplication (of games) preserves identity but not equality.

Properties of order and equality

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Theorem

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Theorem

If $x \geq y$ *and* $y \geq z$, *then* $x \geq z$.

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Theorem

If $x \geq y$ *and* $y \geq z$*, then* $x \geq z$.

Theorem

For any numbers x *and* y*, we have* $x^L < x < x^R$*, and* $x \geq y$ *or* $x \leq y$.

Properties of order and equality

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For any numbers x *and* y*, we have* $x^L < x < x^R$*, and* $x \geq y$ *or* $x \leq y$*.*

Summary

Numbers are totally ordered by $\geq$.

Group properties

Theorem (Properties of addition)

For all x, y, z, *we have* $x + 0 = x$,

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For all x, y, z*, we have* $x + 0 = x$, $x + y \equiv y + x$

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Theorem (Properties of addition)

For all x, y, z, we have $x + 0 = x$, $x + y \equiv y + x$ and $(x + y) + z \equiv x + (y + z)$.

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For all x, y, z, *we have* $x + 0 = x$, $x + y \equiv y + x$ *and* $(x + y) + z \equiv x + (y + z)$.

Theorem (Properties of negation)

For all x, y, *we have* $-(x + y) \equiv -x + -y$,

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Theorem (Properties of addition)

For all x, y, z, *we have* $x + 0 = x$, $x + y \equiv y + x$ *and* $(x + y) + z \equiv x + (y + z)$.

Theorem (Properties of negation)

For all x, y, *we have* $-(x + y) \equiv -x + -y$, $-(-x) \equiv x$ *and* $x + -x = 0$.

Proof.

Induction,

Group properties

Theorem (Properties of addition)

For all x, y, z, *we have* $x + 0 = x$, $x + y \equiv y + x$ *and* $(x + y) + z \equiv x + (y + z)$.

Theorem (Properties of negation)

For all x, y, *we have* $-(x + y) \equiv -x + -y$, $-(-x) \equiv x$ *and* $x + -x = 0$.

Proof.

Induction, left as an exercise to the audience. □

Group properties

Theorem (Properties of addition)

For all x, y, z, *we have* $x + 0 = x$, $x + y \equiv y + x$ *and* $(x + y) + z \equiv x + (y + z)$.

Theorem (Properties of negation)

For all x, y, *we have* $-(x + y) \equiv -x + -y$, $-(-x) \equiv x$ *and* $x + -x = 0$.

Proof.

Induction, left as an exercise to the audience. □

Summary

The operations $+$, $-$ and 0 induce a Group structure on Games.

Ordered group properties

Theorem

For all x, y, z, *we have* $y \geq z$ *iff* $x + y \geq x + z$.

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Numbers form a totally ordered Group.

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Multiplication induces a Ring structure on Numbers and on Games.

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Multiplication induces a Ring structure on Numbers and on Games.

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It is also possible to define division, square roots, etc, turning the Class of numbers into totally ordered Field with many properties.

Real number

How does one single out the real numbers from the Class of numbers?

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Definition

A number x is a real number iff $-n < x < n$ for some integer n and

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- *Dyadic rationals are real numbers.*
- *Real numbers are closed under the field operations.*

Ordinal numbers

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They correspond to the *natural* sum and product, obtained by treating the Cantor normal form of an ordinal as a polynomial.

Algebra and Analysis

Theorem

Every positive number x has a unique positive n^{th} root, for each positive integer n.

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In No, one can define sums indexed over any ordinal.
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One could also use various small subfields of No as a model for non-standard analysis.

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If a and b are integers with b positive, there are unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.

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Almost every number-theoretical problem can be rephrased so as to yield a new problem in Oz, so we get a jackdaw's nest of problems of various kind.

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Almost every number-theoretical problem can be rephrased so as to yield a new problem in Oz, so we get a jackdaw's nest of problems of various kind. Examples include Waring's problem, continued fractions, Pellian equations.

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Our players Left and Right are unwilling to play games that may go on forever (they are both busy people, with heavy political responsibilities).

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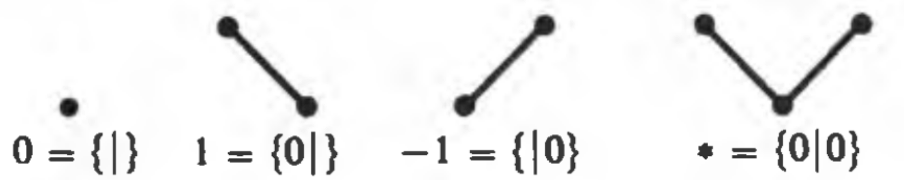

FIG. 4. The simplest games.

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Theorem

Each game belongs to one of the outcome classes above.

Constructions on games

Definition (Negation of a game)

If we reverse the roles of Left and Right in a game G, we obtain the game

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Examples

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We define $G \geq H$ iff $G^R \not\leq$ and $G \not\leq H^L$, $G \leq H$ iff $H \geq G$. We write $G \parallel H$ (G is fuzzy against H) if neither.
We define $G \rhd H$ iff $G \not\leq H$ and $G \lhd H$ iff $G \not\geq H$. We define $<$, $>$ and $=$ as usual.

Interpretation

- $G > H$ iff $G - H$ is won by Left, whoever starts,
- $G < H$ iff $G - H$ is won by Right, whoever starts,
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There has also been some work on Numbers from an algebraic perspective, proving universal properties of the Field / Group of numbers.