Priestley-like duality for subordination lattices

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Introduction

The celebrated Stone duality asserts that the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces and continuous maps. Stone duality was later extended by Priestley in [5]. Indeed, Priestley proved that the category of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category of ordered Stone spaces which are totally order disconnected – the so called Priestley spaces – and order preserving continuous maps between them.

In [2], the authors generalised Stone duality to a modal-like duality for Booleans algebras with a subordination relation $\prec$, interpreted as some kind of strong inclusion. The dual of a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ a subordination, is the space (X, R) obtained by defining the space X as the Stone dual of B and the closed relation R by $x\ R\ y$ iff $\uparrow x \subseteq y$, where $\uparrow x = \{b \in B : a \prec b \text{ for some } a \in x\}$. The dual of the pair (X, R), where X is a Stone space and R a closed relation on X, is the subordination algebra $(B, \prec)$ obtained by defining the algebra B as the dual of X and the subordination $\prec$ by $a \prec b$ iff $R[a] \subseteq b$. These two constructions are each others inverse and they give rise to a duality between the category of subordination algebras and the category of Stone spaces with a closed relation with their respective morphisms. We shall refer to this duality as BBSV duality.

In this paper, we work out the "meet" of Priestley duality and BBSV duality. In other words, we establish a duality between the category whose objects are bounded distributive lattices with a subordination and the category of Priestley spaces with a closed relation satisfying some additional condition. This duality generalises BBSV duality when restricting the distributive lattices to Boolean algebras, and it generalises Priestley duality when restricting the subordinations to trivial ones. This duality was obtained independently by Celani in [4], who gave a different characterisation of the dual spaces. It is thus natural to compare both definitions and to show their equivalence.

We later explore some of the applications of this duality, especially to intuitionistic and modal logic. We also give dual conditions to some common subordination axioms, and do an investigation of subordination lattices similar to the investigation of subordination algebras in [1, Sec. 2]. We also use an extended BBSV duality to get a "classical" duality for subordination lattices, similar to what is done in [1, Sec. 5 & 6].

This paper is organised as follows. In Section 1, we introduce subordinations on bounded distributive lattices, dual relations on the dual Priestley spaces and prove the duality. We also compare our results to those of Celani. In Section 2, we restrict the duality to Heyting algebras and modally definable subordinations. As a corollary, we obtain a duality for modal Heyting algebras, the algebraic structures of intuitionistic modal logic. In Section 3, we give dual conditions to some subordination axioms, including an investigation of lattices subordinations, as in [1]. In Section 4, we work out the "classical" duality arising from an extended BBSV duality. More precisely, we start from an extension of BBSV duality and restrict it successively until we obtain the desired duality for bounded distributive lattices.

1 Bounded distributive lattices and subordinations

In [2], the authors extend Stone duality to Boolean algebras equipped with a subordination. In this section, we use the same method to extend Priestley duality to bounded distributive lattice with a subordination.

Definition 1.1. A *subordination* on a bounded distributive lattice L is a binary relation $\prec$ such that

(S1) $0 \prec 0$ and $1 \prec 1$,
(S2) $a \prec b, c$ implies $a \prec b \wedge c$,
(S3) $a, b \prec c$ implies $a \vee b \prec c$,
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

A lattice equipped with a subordination is called a *subordination lattice*.

Example 1.2. The relation $\prec_1$ defined by $a \prec_1 b$ iff $a = 0$ or $b = 1$ is a subordination.
The order relation $\prec_2 = \leq$ is a subordination.
The total relation $\prec_3$ defined by $a \prec_3 b$ for all a, b is a subordination.

It is well known that the dual of a bounded distributive lattice is a Priestley space. If we wish to extend Priestley's duality, the dual of a subordination lattice should be a Priestley space with some extra structure. In [2], the dual of a subordination on a Boolean algebra is a closed relation on the Stone space. For the distributive lattice case, we need a notion stronger than mere closedness. This motivates the following definition.

Definition 1.3. A *Priestley relation* on a Priestley space $(X, \leq)$ is a binary relation $\sqsubseteq$ such that $x \not\sqsubseteq y$ implies that there is a clopen $\leq$-upset U containing x and a clopen $\leq$-downset V containing y such that $\sqsubseteq[U] \cap V = \emptyset$ (where $\sqsubseteq[U] = \{y \in X : x \sqsubseteq y \text{ for some } x \in U\}$).

A Priestley space equipped with a Priestley relation is called a *Priestley subordination space*.

Example 1.4. The total relation $\sqsubseteq_1$ defined by $x \sqsubseteq_1 y$ for all x, y is a Priestley relation.
The order relation $\sqsubseteq_2 = \leq$ is a Priestley relation.
The empty relation $\sqsubseteq_3$ defined by $x \sqsubseteq_3 y$ for no x, y is a Priestley relation.

We first investigate some properties of Priestley subordination spaces.

Lemma 1.5. *If $(X, \leq, \sqsubseteq)$ is a Priestley subordination space and U a closed set, then $\sqsubseteq[U]$ and $\sqsubseteq^{-1}[U]$ are closed (where $\sqsubseteq^{-1}[U] = \{x \in X : x \sqsubseteq y$ for some $y \in U\}$).*

Proof. It follows from the definition that $\sqsubseteq$ is closed. Indeed, if $(x, y) \notin \sqsubseteq$, then $x \not\sqsubseteq y$ hence there is a clopen upset U containing x and a clopen downset V containing y such that $\sqsubseteq[U] \cap V = \emptyset$. It follows that $(x, y) \in U \times V$ and $U \times V \cap \sqsubseteq = \emptyset$ hence $\sqsubseteq$ is closed. The result then follows from [2, Lem. 2.12]. $\square$

Lemma 1.6. *If $(X, \leq, \sqsubseteq)$ is a Priestley subordination space and $S \subseteq X$, then $\sqsubseteq[S]$ is an upset and $\sqsubseteq^{-1}[S]$ is a downset.*

Proof. We only prove the first claim, the second is similar. Assume that $x \in S$, $x \sqsubseteq y$ and $y \leq z$ but $x \not\sqsubseteq z$. Then there is a clopen upset U and a clopen downset V such that $x \in U$, $z \in V$ and $\sqsubseteq[U] \cap V = \emptyset$. But since V is a downset we have $y \in V$ and since $x \sqsubseteq y$ we have $y \in \sqsubseteq[U]$, which is a contradiction. $\square$

We now compare our definition of Priestley subordination space with that of Celani in [4]. As we will see, both defintions are equivalent.

Definition 1.7. A relation $\sqsubseteq$ on a Priestley space is a *point-closed upset relation* if $\sqsubseteq[x]$ is a closed upset for each x.

Proposition 1.8. *Let $(X, \leq)$ be a Priestley space and $\sqsubseteq$ a binary relation on X. Then the following are equivalent:*

1. *for each closed set U, $\sqsubseteq[U]$ is a closed upset $\sqsubseteq^{-1}[U]$ is a closed downset,*
2. *$\sqsubseteq$ is a point-closed upset relation and $\sqsubseteq^{-1}[U^c]^c$ is an open upset for each clopen upset U,*
3. *$(X, \leq, \sqsubseteq)$ is a Priestley subordination space.*

Proof. $1 \Rightarrow 2$. That $\sqsubseteq$ is a point-closed upset relation follows from the fact that X is Hausdorff, i.e. for every x the singleton $\{x\}$ is closed, hence $\sqsubseteq[x]$ is a closed upset. If U is a clopen upset, then U^c is closed hence $\sqsubseteq^{-1}[U^c]$ is a closed downset. It follows that $\sqsubseteq^{-1}[U^c]^c$ is an open upset.

$2 \Rightarrow 3$. Assume that $x \not\sqsubseteq y$. Then $\sqsubseteq[x]$ is a closed upset not containing y. By the Priestley separation axiom there is a clopen upset U containing $\sqsubseteq[x]$ but not containing y. Then $\sqsubseteq^{-1}[U^c]^c$ is an open upset containing x, hence there is a clopen upset V containing x and contained in $\sqsubseteq^{-1}[U^c]^c$. Then obviously $\sqsubseteq[V] \subseteq U$, $x \in V$ and $y \notin U$ hence $\sqsubseteq$ is a Priestley relation.

$3 \Rightarrow 1$. If $(X, \leq, \sqsubseteq)$ is a Priestley subordination space, then by the two previous lemmas the $\sqsubseteq$ image of a closed upset is a closed upset and the $\sqsubseteq$ inverse image of a closed downset is a closed downset. $\square$

Remark 1.9. The first condition in the previous theorem highlights how the Priestley case is different from the Boolean one. In the Boolean case, we would only require $\sqsubseteq$ to be closed, which would be equivalent to $\sqsubseteq[U]$ and $\sqsubseteq^{-1}[U]$ both being closed if U is closed.

Remark 1.10. It can be shown, using a compactness argument, that if $(X, \leq, \sqsubseteq)$ is a Priestley subordination space, U a closed upset and V a closed downset such that $\sqsubseteq[U] \cap V = \emptyset$, then there is a clopen upset U' and a clopen downset V' such that $U \subseteq U'$, $V \subseteq V'$ and $\sqsubseteq[U'] \cap V' = \emptyset$.

However, this latter condition is not equivalent to the definition of a Priestley relation, as the following example shows.

Example 1.11. Let $(X, \leq)$ be a Priestley space where not every singleton is an upset (that is, there are $x \neq y$ such that $x \leq y$). Let Δ be the diagonal, i.e. the equality relation.

Let us show that Δ satisfies the latter condition. Clearly for any set $S \subseteq X$, $\Delta[S] = S$. If U is a closed upset and V a closed downset such that $\Delta[U] \cap V = \emptyset$, then $U \cap V = \emptyset$. By the Priestley separation axiom, this implies that there is a clopen upset U' such that $U \subseteq U'$ and $U' \cap V = \emptyset$. Let $V' = U'^c$, then V' is a clopen downset such that $V \subseteq V'$ and $U' \cap V' = \emptyset$. We then have $\Delta[U'] \cap V' = \emptyset$, hence Δ satisfies the latter condition.

However, Δ is not a point-closed upset relation, as by the choice of Priestley space there is a point x such that $\Delta[x] = \{x\}$ is not an upset.

1.1 Duality

We now have all the tools at hand to show that there is a correspondence betweeen subordination lattices and Priestley subordination spaces.

From subordinations to Priestley relations For a subordination lattice $(L, \prec)$, let X be the set of prime filters of L ordered by inclusion. For $a \in L$, define

$$\phi(a) = \{x \in X : a \in x\}$$

and topologise X by letting $\{\phi(a), X \setminus \phi(a) : a \in L\}$ be a subbasis for the topology. The resulting space is the Priestley space of L.

Define a relation $\sqsubseteq$ by setting $x \sqsubseteq y$ iff $\twoheaduparrow x \subseteq y$, where $\twoheaduparrow x = \{b \in L : a \prec b \text{ for some } a \in x\}$. Let us check that $\sqsubseteq$ is a Priestley relation. If $x \not\sqsubseteq y$, then there is $a \in \twoheaduparrow x \setminus y$, hence there is $b \in x$ such that $b \prec a$. Then $x \in \phi(b)$ (which is a clopen upset), $y \in X \setminus \phi(a)$ (which is a clopen downset) and $\sqsubseteq[\phi(b)] \subseteq \phi(a)$. Then $(X, \subseteq, \sqsubseteq)$ is the dual of $(L, \prec)$, denoted $(L, \prec)_*$.

From Priestley relations to subordinations For a Priestley subordination space $(X, \leq, \sqsubseteq)$, let L be the set of clopen upsets of X. This is a bounded distributive lattice and is denoted X^*.

Define a subordination $\prec$ by setting $a \prec b$ iff $\sqsubseteq[a] \subseteq b$. It is easy to check that $\prec$ is a subordination on L. The pair $(L, \prec)$ is the dual of $(X, \leq, \sqsubseteq)$, denoted $(X, \leq, \sqsubseteq)^*$.

Example 1.12. Coming back to our examples, $\prec_1$ corresponds to $\sqsubseteq_1$, $\prec_2$ corresponds to $\sqsubseteq_2$ and $\prec_3$ corresponds to $\sqsubseteq_3$.

For any bounded distributive lattice L, ϕ is an isomorphism from L to ${L_*}^*$ and for any Priestley space X, the map $\psi \colon x \mapsto \{a \in X^* : x \in a\}$ is an isomorphism from X to ${X^*}_*$. The two upcoming lemmas show that these map also preserve the extra relation $\prec$ and $\sqsubseteq$, thus making them subordination lattice isomorphism and subordination space isomorphism.

Lemma 1.13. *Let $(L, \prec)$ be a subordination lattice and $\phi \colon L \to {L_*}^*$ the canonical isomorphism. Then $a \prec b$ iff $\phi(a) \prec \phi(b)$.*

Proof. This proof follows closely along the lines of [3, Lem. 3.14].

If $a \prec b$, then $a \in x$ implies $b \in \twoheaduparrow x$ hence $a \in x$ and $\twoheaduparrow x \subseteq y$ implies $b \in y$. This means that $x \in \phi(a)$ and $x \sqsubseteq y$ implies $y \in \phi(b)$, hence $\sqsubseteq[\phi(a)] \subseteq \phi(b)$, that is, $\phi(a) \prec \phi(b)$.

Now suppose that $a \not\prec b$, then $b \notin \twoheaduparrow a$. It is easy to see that $\twoheaduparrow a$ is a filter, therefore, by the ultrafilter theorem, there is an ultrafilter x such that $\twoheaduparrow a \subseteq x$ and $b \notin x$.

Claim. There is a prime filter y such that $a \in y$ and $\twoheaduparrow y \subseteq x$.

Proof of Claim. Let $F = {\uparrow}a$ and $I = L \setminus x$. Then F is a filter containing a and I is an ideal. We show that ${\Uparrow}F \cap I = \emptyset$. If $c \in {\Uparrow}F \cap I$, then $c \in I$ and there is $d \in F$ with $d \prec c$. Therefore $c \notin x$ and $a \leq d \prec c$, thus $c \in {\Uparrow}a$. This yields ${\Uparrow}a \nsubseteq x$, a contradiction.

Consequently, the set Z consisting of the filters G satisfying $a \in G$ and ${\Uparrow}G \subseteq x$ is nonempty because $F \in Z$. It is easy to see that $(Z, \subseteq)$ is an inductive set, hence by Zorn's lemma, Z has a maximal element, say y. We show that y is a prime filter. Suppose $c_1, c_2 \notin y$ and $c_1 \vee c_2 \in y$. Let F_1 be the filter generated by $\{c_1\} \cup y$ and F_2 be the filter generated by $\{c_2\} \cup y$. Since F_1 and F_2 properly contain y, they do not belong to Z, so ${\Uparrow}F_1, {\Uparrow}F_2 \nsubseteq x$. This gives $d_1, d_2 \in y$ and $e \notin x$ such that $c_1 \wedge d_1, c_2 \wedge d_2 \prec e$. By (S3) and distributivity, $(c_1 \vee c_2) \wedge (c_1 \vee d_2) \wedge (d_1 \vee c_2) \wedge (d_1 \vee d_2) \prec e$. But $(c_1 \vee c_2) \wedge (c_1 \vee d_2) \wedge (d_1 \vee c_2) \wedge (d_1 \vee d_2) \in y$, so $e \in {\Uparrow}y \subseteq x$. This is a contradiction, hence y is a prime filter. $\square$

It follows from the Claim that there is $y \in L_*$ such that $y \in \phi(a)$ and $y \sqsubseteq x$. Therefore, $x \in {\sqsubseteq}[\varphi(a)]$. On the other hand, $x \notin \phi(b)$. Thus, ${\sqsubseteq}[\phi(a)] \nsubseteq \phi(b)$, yielding $\phi(a) \not\prec \phi(b)$. $\square$

Lemma 1.14. *Let $(X, \leq, \sqsubseteq)$ be a Priestley subordination space and $\psi \colon X \to X^*{}_*$ the canonical isomorphism. Then $x \sqsubseteq y$ iff $\psi(x) \sqsubseteq \psi(y)$.*

Proof. If $x \sqsubseteq y$, we have $x \in a$ implies $y \in {\sqsubseteq}[a]$ hence $a \in \psi(x)$ and $a \prec b$ implies $b \in \psi(y)$. Clearly this implies $\psi(x) \sqsubseteq \psi(y)$.

Now suppose that $x \not\sqsubseteq y$, then since $\sqsubseteq$ is a Priestley relation there are clopen upsets a and b such that $x \in a$, $y \notin b$ and ${\sqsubseteq}[a] \subseteq b$, or equivalently, $a \in \psi(x)$, $b \notin \psi(y)$ and $a \prec b$. It follows that $\psi(x) \not\sqsubseteq \psi(y)$. $\square$

Let us now define morphisms and prove a duality for those morphisms.

Definition 1.15. A lattice morphism $h \colon K \to L$ between subordination lattices is *monotone* if $a \prec b$ implies $h(a) \prec h(b)$.

Definition 1.16. A continuous map $f \colon X \to Y$ between Priestley subordination spaces is *stable* if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$.

The correspondence extends to morphisms, thus giving a full duality.

If $h \colon K \to L$ is a bounded distributive lattice morphism, then $h_* \colon L_* \to K_* \quad x \mapsto h^{-1}[x]$ is a Priestley morphism. If h is monotone, then h_* is stable.

If $f \colon X \to Y$ is a Priestley morphism, then $f^* \colon Y^* \to X^* \quad a \mapsto f^{-1}[a]$ is a lattice morphism. If f is stable, then f^* is monotone.

Let DSub be the category whose objects are subordination lattices, and whose morphisms are monotone lattice morphisms; and let PrR be the category whose objects are subordination spaces, and whose morphisms are continuous stable maps. Naturality of the correspondence is done in exactly the same way as for Priestley duality. We then obtain the following theorem.

Theorem 1.17. *The category DSub is dually equivalent to the category PrR.*

2 Restrictions of the duality

In this section, we restrict the duality obtained in the previous section to various subcategories of DSub.

2.1 Restriction to modal operators

If $\Box$ is a modal operator on a bounded distributive lattice L, we can define a subordination on L by $a \prec_\Box b$ iff $a \leq \Box b$. The subordinations arising in this way are modally definable.

Definition 2.1. A subordination $\prec$ on a bounded distributive lattice L is *modally definable* if for all $a \in L$, the set $\{b \in L : b \prec a\}$ has a largest element (with respect to the order $\leq$) .

A lattice equipped with a modally definable subordination is called a *modal subordination lattice.*

It is well known that modally definable subordinations and modal operators are equivalent. Indeed, if $\prec$ is a modally definable subordination, define $\Box_\prec a$ to be the largest element of $\{b \in L : b \prec a\}$. Conversely, if $\Box$ is a modal operator, define $\prec_\Box$ by $a \prec_\Box b$ iff $a \leq \Box b$. Then $\Box_\prec$ is a modal operator and we have $\prec_{\Box_\prec} = \prec$ and $\Box_{\prec_\Box} = \Box$.

In the Boolean case, modally definable subordinations can be characterised by their dual relation. We prove that a similar thing happens in the more general case.

Definition 2.2. A Priestley relation $\sqsubseteq$ on a Priestley space $(X, \leq)$ is an *Esakia relation* if $\sqsubseteq^{-1}[U]$ is a clopen $\leq$-downset for all clopen $\leq$-downset U.

Proposition 2.3. *Let $(L, \prec)$ be a subordination lattice and let $(X, \leq, \sqsubseteq)$ be its dual. If $\prec$ is modally definable, then $\sqsubseteq$ is an Esakia relation.*

Let $(X, \leq, \sqsubseteq)$ be a Priestley subordination space let $(L, \prec)$ be its dual. If $\sqsubseteq$ is an Esakia relation, then $\prec$ is modally definable.

Proof. We begin by proving the following claim:

Claim. $\phi(\Box_\prec a) = \sqsubseteq^{-1}[\phi(a)^c]^c$.

Proof of Claim. We have $x \in \sqsubseteq^{-1}[\phi(a)^c]^c$ iff $R[x] \subseteq \phi(a)$ iff $\Uparrow x \subseteq y$ implies $a \in y$ for all $y \in X$. Because $\Uparrow x$ is a filter, by the prime filter theorem, this is in turn equivalent to $a \in \Uparrow x$. Since $\Box_\prec a$ is the largest element of $\{b \in L : b \prec a\}$, we have $a \in \Uparrow x$ iff $\Box_\prec a \in x$, i.e. $x \in \phi(\Box_\prec a)$. $\Box$

Now, take U a clopen downset. Since U^c is a clopen upset, there is $a \in L$ such that $U^c = \phi(a)$. It follows that $\phi(\Box_\prec a) = \sqsubseteq^{-1}[\phi(a)^c]^c = \sqsubseteq^{-1}[U]^c$. Therefore $\sqsubseteq^{-1}[U] = \phi(\Box_\prec a)^c$ is a clopen downset.

For the converse, let U be a clopen upset. We have $V \prec U$ iff $\sqsubseteq[V] \subseteq U$ iff $V \subseteq \sqsubseteq^{-1}[U^c]^c$ and $\sqsubseteq^{-1}[U^c]^c$ is a clopen upset. Therefore $\sqsubseteq^{-1}[U^c]^c$ is the largest element of $\{V \in L : V \prec U\}$ and $\prec$ is modally definable. $\Box$

This already gives us a duality. Define a *modal Priestley space* as a Priestley subordination space $(X, \leq, \sqsubseteq)$ such that $\sqsubseteq$ is an Esakia relation. Let MPS be the full subcategory of PrR whose objects are the modal Priestley spaces; let MDSub be the full subcategory of DSub whose objects are the modal subordination lattices. It follows from the previous result that MDSub is dually equivalent to MPS.

However, one is often more interested in the morphisms that preserve the modal operator. If $(K, \Box)$, $(L, \Box)$ are lattices with a modal operator and $h \colon K \to L$ is a lattice morphism that preserves the modal operator, then h is monotone for $\prec_\Box$. However, if $(K, \prec)$, $(L, \prec)$ are subordination lattices with $\prec$ modally definable and $h \colon K \to L$ is a monotone lattice morphism, it is not always true that h preserves $\Box_\prec$. It is only only if h is strongly monotone.

Definition 2.4. A bounded distributive lattice morphism $h \colon K \to L$ between subordination lattices is *strongly monotone* if it is monotone and $c \prec h(a)$ implies that there is $b \prec a$ with $c \leq h(b)$.

If $h \colon (K, \Box) \to (L, \Box)$ is a lattice morphism that preserves $\Box$, then h is strongly monotone for $\prec_\Box$. Indeed, if $c \prec h(a)$, then $c \leq \Box h(a) = h(\Box a)$ and $\Box a \prec a$. Conversely, if $h \colon (K, \prec) \to (L, \prec)$ is a strongly monotone morphism and the subordinations are modally definable, then the set $\{h(b) : b \prec a\}$ is cofinal in $\{c : c \prec h(a)\}$, hence $h(\Box a) = \Box h(a)$.

Define $\mathsf{DSub}^{\mathsf{st}}$ as the wide subcategory of DSub whose morphisms are the strongly monotone lattice morphisms. In what follows, we give a duality for $\mathsf{DSub}^{\mathsf{st}}$ (note that we are working with general subordinations instead of modally definable ones).

Definition 2.5. A Priestley morphism $f \colon X \to Y$ is *strongly stable* if it is stable and $f(x) \sqsubseteq y$ implies that there is z with $x \sqsubseteq z$ and $f(z) \leq y$.

Proposition 2.6. *If $h \colon K \to L$ is a strongly monotone lattice morphism, then h_* is strongly stable.*

If $f \colon X \to Y$ is a strongly stable Priestley morphism, then f^* is strongly monotone.

Proof. Assume that h is strongly monotone and that $h_*(x) \sqsubseteq y$, that is, $\Uparrow h^{-1}[x] \subseteq y$. Because h is strongly monotone, we have $h^{-1}[\Uparrow x] \subseteq \Uparrow h^{-1}[x]$ hence $h^{-1}[\Uparrow x] \subseteq y$. It follows that $\Uparrow x \cap \downarrow h[y^c] = \emptyset$. By the prime filter theorem, there is a prime filter z such that $\Uparrow x \subseteq z$ and $z \cap h[y^c] = \emptyset$, hence $x \sqsubseteq z$ and $h^{-1}[z] \subseteq y$. Thus h_* is strongly stable.

Now assume that f is strongly stable and that $c \prec f^*[a]$, that is, $\sqsubseteq[c] \subseteq f^{-1}[a]$, or equivalently, $f[\sqsubseteq[c]] \subseteq a$. Since f is strongly stable, we have $\sqsubseteq[f[c]] \subseteq f[\sqsubseteq[c]]$ hence $\sqsubseteq[f[c]] \subseteq a$, i.e. $f[c] \subseteq \sqsubseteq^{-1}[a^c]^c$. As $f[c]$ is a closed upset (c is compact hence so is its image) and $\sqsubseteq^{-1}[a^c]^c$ is an open upset, by the Priestley separation axiom, there is a clopen upset b such that $f[c] \subseteq b$ and $b \subseteq \sqsubseteq^{-1}[a^c]^c$. This b is such that $c \subseteq f^{-1}[b]$ and $\sqsubseteq[b] \subseteq a$, thus f^* is strongly monotone. $\Box$

Let $\mathsf{PrR}^{\mathsf{st}}$ be the wide subcategory of PrR whose morphisms are the continuous strongly stable maps. Then the category $\mathsf{DSub}^{\mathsf{st}}$ is dually equivalent to the category $\mathsf{PrR}^{\mathsf{st}}$.

Now let $\mathsf{MPS}^{\mathsf{st}}$ be the intersection of MPS and $\mathsf{PrR}^{\mathsf{st}}$, i.e. the category whose objects are modal Priestley spaces, and whose morphisms are continuous strongly stable maps; and let $\mathsf{MDSub}^{\mathsf{st}}$ be the intersection of MDSub and $\mathsf{DSub}^{\mathsf{st}}$, i.e. the category whose objects are modal subordination lattices, and whose morphisms are strongly monotone lattice morphisms. It immediately follows from the two previous dualities that $\mathsf{MDSub}^{\mathsf{st}}$ is dually equivalent to $\mathsf{MPS}^{\mathsf{st}}$.

2.2 Restriction to Heyting algebras

If $(L, \prec)$ is a subordination lattice such that L is a Heyting algebra, then obviously its dual $(X, \leq, \sqsubseteq)$ will be a Priestley subordination space where $(X, \leq)$ is an Esakia space (since the construction of $(X, \leq)$ only depends on L and not on $\prec$).

Conversely, if $(X, \leq, \sqsubseteq)$ is a Priestley subordination algebra such that $(X, \leq)$ is an Esakia space, then its dual $(L, \prec)$ is a subordination lattice where L is a Heyting algebra.

As for morphisms, if $h\colon K \to L$ is a monotone Heyting algebra morphism, then its dual $h_*\colon K_* \to L_*$ is a stable Esakia morphism and if $f\colon X \to Y$ is a stable Esakia morphism, then its dual $f^*\colon Y^* \to X^*$ is a monotone Heyting algebra morphism.

Categorically, let HSub be the subcategory of DSub whose objects are Heyting algebras with a subordination, and whose morphisms are the monotone Heyting algebra morphisms; and let EsR be the category whose objects are Esakia spaces with a Priestley relation, and whose morphisms are the continuous stable p-morphisms. Then HSub is dually equivalent to EsR.

We can also combine this duality with the previous one in order to get a duality for intuitionistic modal logic.

Definition 2.7. A Priestley subordination space $(X, \leq, \sqsubseteq)$ is an intuitionistic modal space if $(X, \leq)$ is an Esakia space and $\sqsubseteq$ is an Esakia relation.

A map $f\colon X \to Y$ between two intuitionistic modal spaces is an intuitionistic modal morphism if it is a continuous strongly stable p-morphism.

Now let MHA be the category whose objects are modal Heyting algebras, and whose morphisms are Heyting algebra morphisms that preserve the modal operator; and let IMS be the category whose objects are the intuitionistic modal spaces, and whose morphisms are the intuitionistic modal morphisms. Then obviously MHA is dually equivalent to IMS. Summarising all the dualities in this section, we obtain the theorem below.

Theorem 2.8.
1. *The category* MDSub *is dually equivalent to the category* MPS.
2. *The category* $\mathsf{DSub}^{\mathsf{st}}$ *is dually equivalent to the category* $\mathsf{PrR}^{\mathsf{st}}$.
3. *The category* $\mathsf{MDSub}^{\mathsf{st}}$ *is dually equivalent to the category* $\mathsf{MPS}^{\mathsf{st}}$.
4. *The category* HSub *is dually equivalent to the category* EsR.
5. *The category* MHA *is dually equivalent to the category* IMS.

3 Characterisation of some classes of subordinations

When working with Stone spaces, one may require a subordination to satisfy some extra axioms, making that subordination a de Vries subordination. Most of those axioms can be expressed in the language of bounded distributive lattices, and can be characterised by a condition on the dual relation.

Definition 3.1. Additionally, a subordination may satisfies some of the following extra axioms
(S5) $a \prec b$ implies $a \leq b$,
(S7) $a \prec b$ implies that there is c with $a \prec c \prec b$,
(S8) $a \neq 1$ implies that there is $b \neq 1$ such that $a \prec b$,
(S9) $a \neq 0$ implies that there is $b \neq 0$ such that $b \prec a$.

When working with Stone spaces, (S8) and (S9) are equivalent (provided the other axioms of a de Vries subordination are satisfied), and a subordination satisfies (S8) iff the dual relation is irreducible. That is no longer the case with bounded distributive lattices. We thus need two different dual conditions for those two axioms.

Definition 3.2. Let $(X, \leq, \sqsubseteq)$ be a Priestley subordination space. We say that $\sqsubseteq$ is *forward-irreducible* if the $\sqsubseteq$ image of any proper clopen upset is proper, and *backward-irreducible* if the $\sqsubseteq$ inverse image of any proper clopen downset is proper.

Proposition 3.3. *Let $(L, \prec)$ be a subordination lattice and let $(X, \leq, \sqsubseteq)$ be a Priestley subordination space such that $(L, \prec)$ and $(X, \leq, \sqsubseteq)$ are each other's dual, then*

1. *$\prec$ satisfies (S5) iff $\sqsubseteq$ is reflexive,*
2. *$\prec$ satisfies (S7) iff $\sqsubseteq$ is transitive,*
3. *$\prec$ satisfies (S8) iff $\sqsubseteq$ is forward-irreducible,*
4. *$\prec$ satisfies (S9) iff $\sqsubseteq$ is backward-irreducible.*

Proof. If $\prec$ satisfies (S5), then $\uparrow a \subseteq \uparrow a$ for all a hence $\uparrow x \subseteq x$ for all (prime) filter x. Hence $\sqsubseteq$ is reflexive.

Conversely, if $\sqsubseteq$ is reflexive, then $a \subseteq \sqsubseteq[a]$ for all clopen upset a, hence $a \prec b$ implies $a \subseteq b$ and $\prec$ satisfies (S5).

If $\prec$ satisfies (S7), then $\uparrow x \subseteq \uparrow\uparrow x$. Since $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $\uparrow\uparrow x \subseteq y$, it also implies $x \sqsubseteq z$.

Conversely, if $\sqsubseteq$ is transitive, then $\sqsubseteq[\sqsubseteq[a]] \subseteq \sqsubseteq[a]$. If $a \prec b$, then $\sqsubseteq[\sqsubseteq[a]] \subseteq b$ hence $\sqsubseteq[a] \subseteq \sqsubseteq^{-1}[b^c]^c$. By Lemma 1.5 and Lemma 1.6, $\sqsubseteq[a]$ is a closed upset, $\sqsubseteq^{-1}[b^c]^c$ is an open upset hence by the Priestley separation axiom there is a clopen upset c such that $\sqsubseteq[a] \subseteq c \subseteq \sqsubseteq^{-1}[b^c]^c$. It follows that $\sqsubseteq[a] \subseteq c$ and $\sqsubseteq[c] \subseteq b$ hence $a \prec c \prec b$. Thus $\prec$ satisfies (S7).

If $\prec$ satisfies (S8), let U be a proper clopen upset in X. Since U is a clopen upest, there is $a \in L$ such that $U = \phi(a)$ and since U is proper, we have $a \neq 1$. By (S8), there is $b \neq 1$ such that $a \prec b$, hence $\sqsubseteq[U] \subseteq \phi(b) \subsetneq X$. Hence $\sqsubseteq$ is forward-irreducible.

Conversely, if $\sqsubseteq$ is forward-irreducible, let a be a proper clopen upset. Then $\sqsubseteq[a]$ is a proper closed upset hence there is a proper clopen upset b containing it. Then clearly $b \neq 1$ and $a \prec b$.

The last claim is done similarly. $\square$

3.1 Lattice subordinations

Another nice property that a subordination may have is being a lattice subordination. This is studied extensively in Section 2 of [1]. This subsection is written along the same lines.

Definition 3.4. A subordination $\prec$ on a bounded distributive lattice is a *lattice subordination* if $a \prec b$ implies that there is c such that $c \prec c$ and $a \leq c \leq b$. Obviously a lattice subordination satisfies (S5) and (S7).

A lattice equipped with a lattice subordination is called a *lattice subordination lattice*.

Example 3.5. The subordinations described in Example 1.2 are lattice subordinations.

The results in Section 2 of [1] adapt easily to the bounded distributive lattice case. Let us give some results explicitly.

Lemma 3.6. *Let $\prec$ be a lattice subordination on a bounded distributive lattice L and let $D_\prec = \{a \in D : a \prec a\}$ be the set of reflexive elements of $\prec$. Then $D_\prec$ is a sublattice of D.*

Lemma 3.7. *For a sublattice D of a bounded distributive lattice L, define $\prec_D$ by setting $a \prec_D b$ iff there exists $c \in D$ such that $a \leq c \leq b$. Then $\prec_D$ is a lattice subordination on L.*

Lemma 3.8. *Let L be a bounded distributive lattice.*

1. *If $\prec$ is a lattice subordination on L, then $\prec = \prec_{D_\prec}$.*
2. *If D is a sublattice of L, then $D = D_{\prec_D}$.*

Let DLS be the category whose objects are lattice subordination lattices, and whose morphisms are monotone lattice morphisms; and let DDA be the category whose objects are pairs (L, D) where L is a bounded distributive lattice and D is a sublattice of L, and whose morphisms are lattice morphisms $h\colon L_1 \to L_2$ satisfying $a \in D_1$ implies $h(a) \in D_2$. Then DLS is isomorphic to DDA.

We can also prove a duality for lattice subordination lattices. This is done in a very similar way to [1, Thm. 5.2].

Definition 3.9. A Priestley relation $\sqsubseteq$ on a Priestley space is a *Priestley quasi-order* if $x \not\sqsubseteq y$ implies that there is a clopen $\leq$-upset $\sqsubseteq$-upset U containing x but not y. Equivalently, if U is a closed $\leq$-upset and V a closed $\leq$-downset such that $\sqsubseteq[U] \cap V = \emptyset$, there is a clopen $\leq$-upset $\sqsubseteq$-upset W containing U and disjoint from V.

A Priestley space equipped with a Priestley quasi-order is called a *Priestley quasi-ordered subordination space.*

Example 3.10. The Priestley relations described in Example 1.4 are Priestley quasi-orders.

Proposition 3.11. *Let $(L, \prec)$ be a lattice subordination lattice and let $(X, \leq, \sqsubseteq)$ be its dual. Then $(X, \leq, \sqsubseteq)$ is a Priestley quasi-ordered subordination space.*

Let $(X, \leq, \sqsubseteq)$ be a Priestley quasi-ordered subordination space and let $(L, \prec)$ be its dual. Then $(L, \prec)$ is a lattice subordination lattice.

Proof. For the first claim, assume that $\prec$ is a lattice subordination and let x, y be prime ideals such that $x \not\sqsubseteq y$. Then there are a, b such that $a \in x$, $b \notin y$ and $a \prec b$. Since $\prec$ is a lattice subordination, there is c such that $a \leq c \leq b$ and $c \prec c$. Clearly $\phi(c)$ is a clopen $\leq$-upset containing x but not y. Because $c \prec c$, $z \in \phi(c)$ implies $c \in {\uparrow} z$ hence $\phi(c)$ is also a $\sqsubseteq$-upset. Hence $\sqsubseteq$ is a Priestley quasi-order.

For the second claim, assume that $\sqsubseteq$ is a Priestley quasi-order and let $a \prec b$. Then $\sqsubseteq[a] \cap b^c = \emptyset$ hence by the equivalent statement in the previous definition, there is a clopen $\leq$-upset $\sqsubseteq$-upset c containing a and disjoint from b^c. We thus have $c \prec c$ and $a \leq c \leq b$, hence $\prec$ is a lattice subordination. ☐

Define PrQ as the full subcategory of PrR whose objects are Priestley quasi-ordered subordination spaces. As a result of the previous proposition, we get the subsequent theorem.

Theorem 3.12. *1. The category* DLS *is isomorphic to the category* DDA*.*

2. The category DLS *is dually equivalent to the category* PrQ*.*

3. The category DDA *is dually equivalent to the category* PrQ*.*

4 Classical equivalent to subordination lattices

We have shown that the category of subordination lattices is dually equivalent to the category of Priestley subordination spaces. We now give a classical dual to Priestley subordination spaces. In order to do that, we first establish a duality that is far more general, before restricting to progressively smaller categories.

Priestley subordination spaces are particular instances of Stone spaces with two closed relations. It follows from [2, Thm. 2.22] that those objects correspond to Boolean algebras with two subordinations. If (X, R_1, R_2) is a Stone space with two closed relations, let B be the Boolean algebra of clopen sets of X, and define $U \prec_i V$ iff $R_i[U] \subseteq V$. Then $(B, \prec_1, \prec_2)$ is a Boolean algebra with two subordinations, denoted $(X, R_1, R_2)^*$. Conversely, if $(B, \prec_1, \prec_2)$ is a Boolean algebra with two subordinations, let X be the Stone dual of B and define $x\ R_i\ y$ iff ${\uparrow}_i x \subseteq y$. Then (X, R_1, R_2) is a Stone space with two closed relation, denoted $(B, \prec_1, \prec_2)_*$.

Let StRR be the category whose objects are Stone spaces with two closed relations, and whose morphisms are continuous maps that preserve both relations; and let SubSub be the category whose objects are Boolean algebras with two subordinations, and whose morphisms are Boolean algebra morphisms that preserve both subordinations. Then SubSub is dually equivalent to StRR.

In our study of Priestley subordination spaces, we were interested in triplets (X, R_1, R_2) satisfying some kind of separation axiom.

Definition 4.1. Let X be a Stone space and let R_1, R_2 be two closed relations on X. We say that R_2 is a R_1-*Priestley relation* if $x \not R_2\ y$ implies that there is a clopen R_1-upset U and a clopen R_1-downset V such that $x \in U$, $y \in V$ and $R_2[U] \cap V = \emptyset$.

The dual condition is as follows.

Definition 4.2. Let B be a Boolean algebra and let $\prec_1$, $\prec_2$ be two closed subordinations on B. We say that $\prec_2$ is a $\prec_1$-*compatible subordination* if $a \prec_2 b$ implies that there are $\prec_1$-reflexive elements c, d such that $a \leq c \prec_2 d \leq b$.

Let us show that these two conditions are indeed each other's dual.

Lemma 4.3. *Let $(X, R_1, R_2) \in \mathsf{SubSub}$ and let $(B, \prec_1, \prec_2)$ be its dual. If R_2 is R_1-Priestley, then $\prec_2$ is $\prec_1$-compatible.*

Proof. Let U, V be clopen sets such that $U \prec_2 V$, that is, $R_2[U] \subseteq V$. Then for any $x \in U$, $y \in V^c$, we have $x \not R_2\, y$ hence there is a clopen R_1-upset $U_{x,y}$ and a clopen R_1-downset $V_{x,y}$ such that $x \in U_{x,y}$, $y \in V_{x,y}$ and $R_2[U_{x,y}] \cap V_{x,y} = \emptyset$. Now fix $y \in V^c$, we have $U \subseteq \bigcup_{x \in U} U_{x,y}$ hence by compactness there are $x_1, \ldots, x_n \in U$ such that $U \subseteq U_{x_1,y} \cup \cdots \cup U_{x_n,y}$. Let $U_y = U_{x_1,y} \cup \cdots \cup U_{x_n,y}$ and $V_y = V_{x_1,y} \cap \cdots \cap V_{x_n,y}$. Then $U \subseteq U_y$, $y \in V_y$ and $R_2[U_y] \cap V_y = \emptyset$. We have $V^c \subseteq \bigcup_{y \in V^c} V_y$ hence by compactness there are $y_1, \ldots, y_m \in V^c$ such that $V^c \subseteq V_{y_1} \cup \cdots \cup V_{y_m}$. Let $U' = U_{y_1} \cap \cdots \cap U_{y_m}$ and $V' = V_{y_1} \cup \cdots \cup V_{y_m}$. Then $U \subseteq U'$, $V'^c \subseteq V$ and $R_2[U'] \cap V' = \emptyset$, that is, $U \leq U' \prec_2 V'^c \leq V$. Furthermore, U' and V'^c are clopen R_1-upsets hence they are $\prec_1$-reflexive. □

Lemma 4.4. *Let $(B, \prec_1, \prec_2) \in \mathsf{StRR}$ and let (X, R_1, R_2) be its dual. If $\prec_2$ is $\prec_1$-compatible, then R_2 is R_1-Priestley.*

Proof. Let x,y be ultrafilters such that $x \not R_2\, y$, that is, $\uparrow_2 x \not\subseteq y$. Then there are elements a, b such that $a \prec_2 b$, $a \in x$ and $b \notin y$. Because $\prec_2$ is $\prec_1$ compatible, there are $\prec_1$-reflexive elements c, d such that $a \leq c \prec_2 d \leq b$. We then have $x \in \phi(c)$, $R_2[\phi(c)] \subseteq \phi(d)$ and $y \notin \phi(d)$. Because c, d are $\prec_1$-reflexive, $\phi(c)$ is a clopen R_1-upset containing x and $\phi(d)^c$ is a clopen R_1-upset containing y. Furthermore $R_2[\phi(c)] \cap \phi(d)^c = \emptyset$. □

Categorically, let StRP be the full subcategory of StRR whose objects are the objects $(X, R_1, R_2) \in \mathsf{StRR}$ such that R_2 is R_1-Priestley, and let SubCom be the full subcategory of SubSub whose objects are the objects $(B, \prec_1, \prec_2) \in \mathsf{SubSub}$ such that $\prec_2$ is $\prec_1$-compatible. Then SubCom is dually equivalent to StRP.

We still have to formulate a condition for the relation R_1. We borrow these conditions from [1].

Definition 4.5. A closed relation $\leq$ on a Stone space is a *Priestley quasi-order* if $x \not\leq y$ implies that there is a clopen upset U with $x \in U$, $y \notin U$.

Definition 4.6. A subordination $\prec$ on a Boolean algebra is a *lattice subordination* if $a \prec b$ implies that there is a reflexive c with $a \leq c \leq b$.

Remark 4.7. A relation $\leq$ on a Stone space is a Priestley quasi-order iff it is $\leq$-Priestley and reflexive. Indeed, if $\leq$ is a Priestley quasi-order and $x \not\leq y$, then there is a clopen upset U with $x \in U$ and $y \notin U$. Then U^c is a clopen downset with $y \in U^c$ and $\leq[U] \cap U^c = \emptyset$, hence $\leq$ is $\leq$-Priestley. To show that $\leq$ is reflexive, assume that $x \not\leq x$. Then there is a clopen upset U with $x \in U$ and $x \notin U$, absurd! Conversely, assume that $\leq$ is $\leq$-Priestley and reflexive. If $x \not\leq y$, then there is a clopen upset U and a clopen downset V with $x \in U$, $y \in V$ and $\leq[U] \cap V = \emptyset$. Because $\leq$ is reflexive, we have $U = \leq[U]$ hence $y \in V \subseteq \leq[U]^c = U^c$. Hence U is a clopen upset with $x \in U$ and $y \notin U$.

A subordination $\prec$ on a Boolean algebra is a lattice subordination iff it is $\prec$-compatible and satisfies (S5). Indeed, if $\prec$ is a lattice subordination and $a \prec b$, then there is a reflexive element c with $a \leq c \leq b$. Then c, c are reflexive elements with $a \leq c \prec c \leq b$. To show that $\prec$ is (S5), assume that $a \prec b$. Then there is a reflexive element c with $a \leq c \leq b$, hence $a \leq b$. Conversely, assume that $\prec$ is $\prec$-compatible and (S5). If $a \prec b$, then there are reflexive elements c, d such that $a \leq c \prec d \leq b$. Because $\prec$ is (S5), we have $c \leq b$ hence c is a reflexive element with $a \leq c \leq b$.

It follows from [2, Lem. 4.6(1)] that if $(B, \prec_1, \prec_2)$ and (X, R_1, R_2) are each other's dual, then $\prec_1$ is (S5) iff R_1 is reflexive. It follows from the two previous lemmas that $\prec_1$ is $\prec_1$-compatible iff R_1 is R_1-Priestley. Hence from the previous remark, it follows that $\prec_1$ is a lattice subordination iff R_1 is a Priestley quasi-order, as was obtained in [1, Cor. 5.3].

Let StQR be the full subcategory of StRR whose objects are the objects $(X, R_1, R_2) \in \mathsf{StRR}$ such that R_1 is a Priestley quasi-order; and let LatSub be the full subcategory of SubSub whose objects are the objects $(B, \prec_1, \prec_2) \in \mathsf{SubSub}$ such that $\prec_1$ is a lattice subordination. Then LatSub is dually equivalent to StQR.

Now let StQP be the intersection of StRP and StQR, i.e. the category whose objects are $(X, R_1, R_2) \in \mathsf{StRR}$ such that R_1 is a Priestley quasi-order and R_2 is R_1-Priestley; and let LatCom be the intersection of SubCom and LatSub, i.e. the category whose objects are the objects $(B, \prec_1, \prec_2) \in \mathsf{SubSub}$ such that $\prec_1$ is a lattice subordination and $\prec_2$ is $\prec_1$-compatible. Then LatCom is dually equivalent to StQP.

As was shown in [1, Thm. 2.10], lattice subordinations on a Boolean algebra correspond to sublattices of that algebra and vice versa. To be able to give a similar correspondence for LatSub, we need to define a notion of compatibility with respect to a sublattice.

Definition 4.8. Let B be a Boolean algebra, D a sublattice of B and $\prec$ a subordination on B. We say that $\prec$ is a D-compatible subordination if $a \prec b$ implies that there are $c, d \in D$ such that $a \leq c \prec d \leq b$.

Remark 4.9. Given a Boolean algebra B, a sublattice D of B and a subordination $\prec$ on D, there is a unique D-compatible extension $\prec_B$ of $\prec$ to B, defined as $a \prec_B b$ iff there are $c, d \in D$ with $a \leq c \prec d \leq b$.

Given a subordination $\prec$ on a Boolean algebra B, define $D_\prec = \{a \in B : a \prec a\}$ as the set of reflexive elements. Then $D_\prec$ is a sublattice of B. Conversely, given a sublattice D of a Boolean algebra B, define $a \prec_D b$ iff there is $c \in D$ such that $a \leq c \leq b$. Then $\prec_D$ is a subordination on B. Furthermore $\prec_{D_\prec} = \prec$ and $D_{\prec_D} = D$. It is also straightforward to check that a subordination is D-compatible iff it is $\prec_D$-compatible.

Let BDCom be the category whose objects are triplets $(B, D, \prec)$ where B is a Boolean algebra, D a sublattice of B and $\prec$ a D-compatible subordination on B, and whose morphisms are the Boolean algebra morphisms that restrict to lattice morphisms between the sublattices and preserve $\prec$. Then BDCom is isomorphic to LatCom.

It follows that BDCom is dually equivalent to StQP. In fact, this duality can be obtained directly. If $(B, D, \prec) \in \mathsf{BDCom}$, let X be the Stone space of B, define $x \leq y$ iff $x \cap D \subseteq y$, define $x \sqsubseteq y$ iff $\uparrow x \subseteq y$ and let $(B, D, \prec)_* = (X, \leq, \sqsubseteq)$. Conversely, if $(X, \leq, \sqsubseteq) \in \mathsf{StQP}$, let B be the Boolean algebra of clopen sets of X, D the sublattice of clopen $\leq$-upsets, define $U \prec V$ iff $\sqsubseteq[U] \subseteq V$ and let $(X, \leq, \sqsubseteq)^* = (B, D, \prec)$.

Lemma 4.10. *Let $(B, D, \prec) \in \mathsf{BDCom}$. Then $(X, \leq, \sqsubseteq) = (B, D, \prec)_* \in \mathsf{StQP}$.*

Proof. Clearly X is a Stone space. We show that $\leq$ is a Priestley quasi-order. Assume that $x \not\leq y$, that is, there is $a \in D$ with $a \in x$ and $a \notin y$. Then $\phi(a)$ is a clopen $\leq$-upset with $x \in \phi(a)$, $y \notin \phi(a)$.

Let us now show that $\sqsubseteq$ is $\leq$-Priestley. Assume that $x \not\sqsubseteq y$, that is, there are $a \prec b$ with $a \in x$ and $b \notin y$. Because $\prec$ is D-compatible, there are $c, d \in D$ with $a \leq c \prec d \leq b$. Then clearly $x \in \phi(c)$, $y \in \phi(d)^c$ and $\sqsubseteq[\phi(c) \cap \phi(d)^c = \emptyset$. Because $c, d \in D$, $\phi(c)$ is a clopen $\leq$-upset and $\phi(d)^c$ is a clopen $\leq$-downset. ☐

Lemma 4.11. *Let $(X, \leq, \sqsubseteq) \in \mathsf{StQP}$. Then $(B, D, \prec) = (X, \leq, \sqsubseteq)^* \in \mathsf{BDCom}$.*

Proof. Clearly B is a Boolean algebra and D is a sublattice of B. It is also straightforward to check that $\prec$ is a subordination. Let us show that $\prec$ is D-compatible. Assume that $U \prec V$, that is, $\sqsubseteq[U] \subseteq V$. Then for any $x \in U$, $y \notin V$, we have $x \not\sqsubseteq y$ hence there is a clopen $\leq$-upset $U_{x,y}$ and a clopen $\leq$-downset $V_{x,y}$ such that $x \in U_{x,y}$, $y \in V_{x,y}$ and $\sqsubseteq[U_{x,y}] \cap V_{x,y} = \emptyset$. Fix $y \notin V$, then $U \subseteq \bigcup_{x \in U} U_{x,y}$ hence by compactness there are $x_1, \dots, x_m \in U$ such that $U \subseteq U_{x_1,y} \cup \dots \cup U_{x_m,y}$. Let $U_y = U_{x_1,y} \cup \dots \cup U_{x_m,y}$ and $V_y = V_{x_1,y} \cup \dots \cup V_{x_m,y}$. Then for all $y \in V^c$, $U \subseteq U_y$, $y \in V_y$ and $\sqsubseteq[U_y] \cap V_y = \emptyset$. We have $V^c \subseteq \bigcup_{y \in V^c} V_y$ hence by compactness there are $y_1, \dots, y_n \in V^c$ such that $V^c \subseteq V_{y_1} \cup \dots \cup V_{y_n}$. Let $U' = U_{y_1} \cap \dots \cap U_{y_n}$. Then $U \subseteq U'$, $V^c \subseteq V'$ and $\sqsubseteq[U'] \cap V' = \emptyset$. It follows that $U \leq U' \prec V'^c \leq V$, with $U', V'^c \in D$. ☐

Let $(X, \leq, \sqsubseteq) \in \mathsf{StQP}$ and $(B, D, \prec) \in \mathsf{BDCom}$ be each other's dual. By [1, Lem. 6.4], we know that $\leq$ is a partial order iff B is generated by D. Letting GBDCom be the full subcategory of BDCom whose objects are the objects $(B, D, \prec) \in \mathsf{BDCom}$ such that D generates B, we get that GBDCom is dually equivalent to PrR.

Since DSub is dually equivalent to PrR, it follows that DSub and GBDCom are equivalent. The equivalence can also be obtained directly. The functor $\mathfrak{U}\colon \mathsf{BDCom} \to \mathsf{DSub}$ sending each $(B, D, \prec)$ to $(D, \prec)$ has a left adjoint $\mathfrak{G}\colon \mathsf{DSub} \to \mathsf{GBDCom}$ sending each $(D, \prec)$ to $(B(D), D, \prec_B)$ where $B(D)$ is the free Boolean extension of D and $\prec_B$ is the unique D-compatible extension of $\prec$ to $B(D)$ described in 4.9. If $(D, \prec) \in \mathsf{DSub}$, then $(B(D), D, \prec_B) \in \mathsf{GBDCom}$, therefore GBDCom is equivalent to DSub. In summary, we get the theorem hereunder.

Theorem 4.12. 1. *The category SubCom is dually equivalent to the category StRP.*
2. *The category LatSub is dually equivalent to the category StQR.*
3. *The category LatCom is dually equivalent to the category StQP.*

4. The category BDCom *is isomorphic to the category* LatCom*.*
5. The category BDCom *is dually equivalent to the category* StQP*.*
6. The category GBDCom *is dually equivalent to the category* PrR*.*
7. The category GBDCom *is equivalent to the category* DSub*.*

We conclude this paper with five tables. The first four contain lists of the categories considered in this paper. The fifth table summarises the obtained isomorphisms, equivalences and dualities, together with the corresponding theorem numbers. For two categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, $\mathcal{C} \sim \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent, and $\mathcal{C} \overset{\mathrm{d}}{\sim} \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are dually equivalent.

Category	Objects (and morphisms)
DSub	Subordination lattices
$\mathsf{DSub}^{\mathsf{st}}$	Subordination lattices (and strong morphisms)
MDSub	Modal subordination lattices
$\mathsf{MDSub}^{\mathsf{st}}$	Modal subordination lattices (and strong morphisms)
HSub	Heyting algebras with a subordination
MHA	Modal Heyting algebras
DLS	Lattice subordination lattices
DDA	Distributive lattices with a sublattice

Table 1: Categories of distributive lattices with subordination

Category	Objects
SubSub	Boolean algebras with two subordinations
SubCom	Boolean algebras with an arbitrary subordination and a compatible subordination
LatSub	Boolean algebras with a lattice subordination and an arbitrary subordination
LatCom	Boolean algebras with a lattice subordination and a compatible subordination
BDCom	Boolean algebras with a sublattice and a compatible subordination
GBDCom	Boolean algebras with a generating sublattice and a compatible subordination

Table 2: Categories of Boolean algebras with two subordinations

Category	Objects (and morphisms)
PrR	Priestley subordination spaces (Priestley space with a Priestley relation)
$\mathsf{PrR}^{\mathsf{st}}$	Priestley subordination spaces (and strong morphisms)
MPS	Modal Priestley space (Priestley space with an Esakia relation)
$\mathsf{MPS}^{\mathsf{st}}$	Modal Priestley space (and strong morphisms)
EsR	Esakia space with a Priestley relation
IMS	Intuitionistic modal space (strong morphisms)
PrQ	Priestley quasi-ordered subordination spaces

Table 3: Category of Priestley spaces with a relation

Category	Objects
StRR	Stone spaces with two closed relation
StRP	Stone spaces with a closed relation and a Priestley relation
StQR	Stone space with a Priestley quasi-order and a closed relation
StQP	Stone space with a Priestley quasi-order and a Priestley relation

Table 4: Category of Stone spaces with two relations

DSub	$\sim$	GBDCom	$\overset{\mathrm{d}}{\sim}$	PrR	Thm. 1.17, 4.12
MDSub	$\overset{\mathrm{d}}{\sim}$	MPS			Thm. 2.8
$\mathsf{DSub}^{\mathsf{st}}$	$\overset{\mathrm{d}}{\sim}$	$\mathsf{PrR}^{\mathsf{st}}$			Thm. 2.8
$\mathsf{MDSub}^{\mathsf{st}}$	$\overset{\mathrm{d}}{\sim}$	$\mathsf{MPS}^{\mathsf{st}}$			Thm. 2.8
HSub	$\overset{\mathrm{d}}{\sim}$	EsR			Thm. 2.8
MHA	$\overset{\mathrm{d}}{\sim}$	IMS			Thm. 2.8
DLS	$\cong$	DDA	$\overset{\mathrm{d}}{\sim}$	PrQ	Thm. 3.12
SubSub	$\overset{\mathrm{d}}{\sim}$	StRR			Sec. 4, third paragraph
SubCom	$\overset{\mathrm{d}}{\sim}$	StRP			Thm. 4.12
LatSub	$\overset{\mathrm{d}}{\sim}$	StQR			Thm. 4.12
LatCom	$\cong$	BDCom	$\overset{\mathrm{d}}{\sim}$	StQP	Thm. 4.12

Table 5: Isomorphisms, equivalences and dualities

References

[1] Guram Bezhanishvili. ‘Lattice subordinations and Priestley duality’. In: *Algebra Universalis* 70.4 (Dec. 2013), pp. 359–377. ISSN: 00025240. DOI: 10.1007/s00012-013-0253-0.

[2] Guram Bezhanishvili, Nick Bezhanishvili, Sumit Sourabh and Yde Venema. ‘Irreducible equivalence relations, Gleason spaces, and de Vries duality’. In: *Applied Categorical Structures* 25.3 (June 2017), pp. 381–401. ISSN: 15729095. DOI: 10.1007/s10485-016-9434-2.

[3] Guram Bezhanishvili, Nick Bezhanishvili, Sumit Sourabh and Yde Venema. ‘Subordinations, closed relations, and compact Hausdorff spaces’. In: *ILLC Prebublication Series, PP-2014-23* (2014). URL: https://eprints.illc.uva.nl/id/document/1242.

[4] Sergio A. Celani. ‘Subordinations on Bounded Distributive Lattices’. In: *Order* (Feb. 2022), pp. 1–27. ISSN: 15729273. DOI: 10.1007/s11083-021-09580-5.

[5] H. A. Priestley. ‘Representation of Distributive Lattices by means of ordered Stone Spaces’. In: *Bulletin of the London Mathematical Society* 2.2 (July 1970), pp. 186–190. ISSN: 1469-2120. DOI: 10.1112/BLMS/2.2.186.